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Stability in delay difference equations with nonuniform exponential behavior ☆

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Abstract

We study the stability under perturbations for delay difference equations in Banach spaces. Namely, we establish the (nonuniform) stability of linear *nonuniform* exponential contractions under sufficiently small perturbations. We also obtain a stable manifold theorem for perturbations of linear delay difference equations admitting a nonuniform exponential dichotomy, and show that the stable manifolds are Lipschitz in the perturbation.

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1. Introduction

Our main objective is to study the stability under perturbations of linear *delay difference equations* that possesses some type of *nonuniform* exponential behavior. Namely, we consider nonuniform exponential *contractions* and nonuniform exponential *dichotomies* and show that in both cases there is (nonuniform) stability under sufficiently small perturbations. These are in fact

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the weakest possible assumptions under which it is possible to establish stability. We refer to [2] for a detailed related discussion in the case of ordinary differential equations.

More precisely, we consider the delay equation

$$x(m+1) = L_m x_m, \quad m \geqslant 1, \tag{1}$$

for some linear operators $L_m : X \to Y$, where X is the space of functions $\phi : \{r, r+1, \dots, 0\} \to Y$ with values in some Banach space Y for some integer $r < 0$. The function $x_m \in X$ in (1) is defined by $x_m(j) = x(m+j)$. For example, when $Y = \mathbb{R}^k$ each operator L_m can be written in the form

$$L_m \phi = \sum_{j=r}^{0} \eta(m, j)\phi(j)$$

for some (uniquely determined) $k \times k$ matrices $\eta(m, j) : \mathbb{R}^k \to \mathbb{R}^k$, and thus Eq. (1) can be written in the form

$$x(m+1) = \sum_{j=r}^{0} \eta(m, j) x(m+j), \quad m \geqslant 1.$$

When Eq. (1) admits a nonuniform exponential contraction we show that for sufficiently small perturbations $f_m : X \to Y$ (see (8) below) the origin in the nonlinear delay equation

$$x(m+1) = L_m x_m + f_m(x_m)$$

is still exponentially stable (see Theorem 1), for initial conditions which may need to be taken exponentially small in the initial time (although with small exponentials when compared to the negative Lyapunov exponents in the nonuniform contraction).

We also consider the case when Eq. (1) admits a nonuniform exponential dichotomy and we establish a stable manifold theorem under sufficiently small perturbations (see Theorem 2). We also show that the stable manifolds are Lipschitz in the perturbation (see Theorem 3).

In the special case of *uniform* exponential behavior we refer to [3] (see also the references therein), for a treatment which is related in spirit to ours. Nevertheless, we emphasize that due to the nonuniform behavior we are not able to use the same techniques. We refer to [1] for a detailed presentation of the core of the nonuniform hyperbolicity theory.

2. Stability of nonuniform contractions

We obtain here the (nonuniform) exponential stability of the zero solution of a delay difference equation that is a perturbation of a linear nonuniform exponential contraction.

2.1. Basic setup

For simplicity of the notation, we will always denote by $[m, n]$, $(-\infty, n]$ and $[n, +\infty)$ respectively the sets $[m, n] \cap \mathbb{Z}$, $(-\infty, m] \cap \mathbb{Z}$ and $[n, +\infty) \cap \mathbb{Z}$. Fix $r \in \mathbb{Z}_0^-$ (this is the delay). Consider a Banach space Y, and let X be the space of functions $\phi : [r, 0] \to Y$ with the norm

$$\|\phi\| = \max\{|\phi(j)| : j \in [r, 0]\}, \tag{2}$$

where $|\cdot|$ is the norm in Y. For any function $x:(-\infty,m]\to Y$ and $n\leqslant m$, we define $x_n\in X$ by $x_n(j)=x(n+j)$ for $j\in[r,0]$.

Given linear operators $L_m:X\to Y$ for $m\in\mathbb{N}$, we consider the dynamics defined by

$$x(m+1)=L_m x_m. \tag{3}$$

For each $n\in\mathbb{N}$ and $\phi\in X$, we obtain a unique function $x:[n+r,+\infty)\to Y$, denoted by $x(\cdot,n,\phi)$, such that $x_n=\phi$ and (3) holds for all $m\geqslant n$. For each $m\geqslant n$ we define the operator $T(m,n)$ on X by

$$T(m,n)\phi=x_m(\cdot,n,\phi),\quad \phi\in X. \tag{4}$$

Clearly, $T(m,n)$ is linear, $T(m,m)=\mathrm{Id}$, and

$$T(l,m)T(m,n)=T(l,n),\quad l\geqslant m\geqslant n.$$

2.2. *Nonuniform exponential contractions*

We say that Eq. (3) admits a *nonuniform exponential contraction* if there exist constants $a<0$, $D\geqslant 1$ and $\varepsilon\geqslant 0$ such that for $m\geqslant n$ we have

$$\left\|T(m,n)\right\|\leqslant De^{a(m-n)+\varepsilon n}. \tag{5}$$

We give a simple example of a nonuniform exponential contraction when $Y=\mathbb{R}$. Set $r=0$. Given $\omega>\alpha>0$, we let

$$\eta(m,0)=e^{-\omega+\alpha m\cos(\pi m)-\alpha(m-1)\cos(\pi(m-1))+\alpha\sin(\pi m)-\alpha\sin(\pi(m-1))},$$

in which case

$$L_m\phi=e^{-\omega+\alpha m\cos(\pi m)-\alpha(m-1)\cos(\pi(m-1))+\alpha\sin(\pi m)-\alpha\sin(\pi(m-1))}\phi(0). \tag{6}$$

Thus Eq. (3) becomes $x(m+1)=A_m x(m)$ where A_m is the exponential in (6). Notice that

$$\begin{aligned}T(m,n)&=A_{m-1}\cdots A_n\\&=e^{(-\omega+\alpha)(m-n)+\alpha m(\cos(\pi m)-1)-\alpha n(\cos(\pi n)-1)+\alpha(\sin(\pi n)-\sin(\pi m))},\end{aligned}$$

and hence

$$T(m,n)\leqslant e^{2\alpha}e^{(-\omega+\alpha)(m-n)+2\alpha n}.$$

This establishes (5) with $a=-\omega+\alpha$ and $\varepsilon=2\alpha$. Furthermore, for $m=4k$ and $n=3k$ with $k\in\mathbb{N}$ we have

$$T(m,n)=e^{(-\omega+\alpha)(m-n)+2\alpha n},$$

and thus it is impossible to take $\varepsilon=0$ in (5) (in other words the contraction is not uniform).

2.3. Stability of nonuniform exponential contractions

We now consider the delay difference equation

$$x(m+1) = L_m x_m + f_m(x_m) \tag{7}$$

for some linear operators L_m as above, and some functions $f_m : X \to Y$ for $m \in \mathbb{N}$. We assume that $f_m(0) = 0$ for each $m \in \mathbb{N}$, and that there exist constants $\delta, q > 0$ (independent of m) such that

$$\left|f_m(u) - f_m(v)\right| \leqslant \delta \|u - v\| \left(\|u\|^q + \|v\|^q\right), \tag{8}$$

for every $m \in \mathbb{N}$ and $u, v \in X$.

The following is our stability result.

Theorem 1. *If Eq.* (3) *admits a nonuniform exponential contraction and $qa + \varepsilon < 0$, then for every sufficiently small $\delta > 0$, the solution of* (7) *with initial condition $(n, \phi) \in \mathbb{N} \times X$ and $\|\phi\| \leqslant e^{-\varepsilon(1+2/q)n}$ satisfies*

$$\|x_m\| \leqslant 2De^{a(m-n)+\varepsilon n}\|\phi\| \quad \textit{for every } m \geqslant n. \tag{9}$$

Proof. We start with an auxiliary result.

Lemma 1. *The solution $x = x(\cdot, n, \phi)$ of* (7) *satisfies the equation*

$$x_m = T(m, n)\phi + \sum_{j=n}^{m-1} T(m, j+1)\bigl(\Gamma f_j(x_j)\bigr), \quad m \geqslant n, \tag{10}$$

where each $\Gamma(l)$ is a $k \times k$ matrix with $\Gamma(0) = \mathrm{Id}$ and $\Gamma(l) = 0$ for $l \in [r, 0)$.

Although the statement in Lemma 1 is well known to the experts, we have been unable to find an appropriate reference. Thus, for completeness we give a simple proof. It should be noted that the symbol $\Gamma f_j(x_j)$ in (10) denotes the function in X defined by $[r, 0] \ni l \mapsto \Gamma(l) f_j(x_j)$.

Proof of Lemma 1. We first observe that

$$x_{m+1} = T(m+1, m)x_m + \Gamma f_m(x_m). \tag{11}$$

By (4) we have

$$\bigl(T(m+1, m)\phi\bigr)(0) = x_{m+1}(0, m, \phi) = x(m+1, m, \phi) = L_m\phi.$$

Therefore,

$$\begin{aligned}\bigl[T(m+1, m)x_m + \Gamma f_m(x_m)\bigr](0) &= L_m x_m + f_m(x_m) \\ &= x(m+1) = x_{m+1}(0).\end{aligned}$$

For each $l \in [r, 0)$ we have

$$\big[T(m+1,m)x_m + \Gamma f_m(x_m)\big](l) = T(m+1,m)x_m(l) = x_m(l+1).$$

The last equality follows from the fact that the values of $T(m+1,m)x_m$ obtained from Eq. (3) on $[r,0)$ are in fact $x_m(r+1), \dots, x_m(-1)$, and thus coincide with those obtained from Eq. (7) at those points. This establishes (11). We now proceed by induction in $m \geqslant n$. Clearly, (10) holds for $m = n$ (by convention of the sum). Assuming that it holds for some m, by (11) and the induction hypothesis we obtain

$$\begin{aligned} x_{m+1} &= T(m+1,m)x_m + \Gamma f_m(x_m) \\ &= T(m+1,m)\left(T(m,n)\phi + \sum_{j=n}^{m-1} T(m,j+1)\big(\Gamma f_j(x_j)\big)\right) + \Gamma f_m(x_m) \\ &= T(m+1,n)\phi + \sum_{j=n}^{m} T(m+1,j+1)\big(\Gamma f_j(x_j)\big). \end{aligned}$$

This establishes (10). □

We proceed with the proof of the theorem. In view of (10), we consider the operator R defined by

$$(Rx)_m = T(m,n)\phi + \sum_{k=n}^{m-1} T(m,k+1)\big(\Gamma f_k(x_k)\big), \quad m \geqslant n, \tag{12}$$

in the space

$$\mathcal{C} = \big\{x : [n+r, +\infty) \to Y \colon \|x\| \leqslant 2De^{ar}\|\phi\|\big\},$$

with the norm

$$\|x\| = \sup\big\{\|x_m\|e^{-\gamma(m,n)} \colon m \geqslant n\big\}, \quad \gamma(m,n) = a(m-n) + \varepsilon n. \tag{13}$$

One can easily verify that $\mathcal{C}$ is a complete metric space.

Note that $T(m,n)\phi(j) = T(m+j,n)\phi(0)$ for $j \in [r,0]$. Therefore, for each $j \in [r,0]$ it follows from (12) that

$$(Rx)(m+j) = \big[T(m+j,n)\phi\big](0) + \sum_{k=n}^{m-1} T(m+j,k+1)\big(\Gamma f_k(x_k)\big)(0), \tag{14}$$

and thus

$$\big|(Rx)(m+j)\big| \leqslant \big\|T(m+j,n)\phi\big\| + \sum_{k=n}^{m+j-1} \big\|T(m+j,k+1)\big\| \cdot \big|f_k(x_k)\big|.$$

By (5) and (8) we obtain

$$\left|(Rx)(m+j)\right| \leqslant De^{a(m+j-n)+\varepsilon n}\|\phi\| + \sum_{k=n}^{m-1} D\delta e^{a(m+j-k-1)+\varepsilon(k+1)}\|x_k\|^{q+1}.$$

Furthermore, since

$$\|x_k\| \leqslant 2De^{ar}e^{\gamma(k,n)}\|\phi\| \quad \text{and} \quad \|\phi\| \leqslant e^{-\varepsilon(1+2/q)n},$$

we have

$$\begin{aligned}
\left|(Rx)(m+j)\right| &\leqslant De^{a(m+j-n)+\varepsilon n}\|\phi\| \\
&\quad + \left(2De^{ar}\right)^{q+1} D\delta\|\phi\| \sum_{k=n}^{m-1} e^{a(m+j-k-1)+\varepsilon(k+1)+(q+1)a(k-n)-\varepsilon n} \\
&\leqslant De^{\gamma(m,n)}e^{aj}\|\phi\| \\
&\quad + \left(2De^{ar}\right)^{q+1} D\delta\|\phi\| e^{a(m+j-n-1)}e^{\varepsilon} \sum_{k=n}^{m-1} e^{(qa+\varepsilon)(k-n)} \\
&\leqslant De^{\gamma(m,n)}e^{aj}\|\phi\| \\
&\quad + \left(2De^{ar}\right)^{q+1} D\delta\|\phi\| \frac{1}{1-e^{qa+\varepsilon}} e^{a(j-1)+\varepsilon}e^{\gamma(m,n)},
\end{aligned}$$

using also the condition $qa+\varepsilon<0$. Therefore, since $j \in [r,0]$ we obtain

$$\left|(Rx)(m+j)\right| \leqslant De^{ar}e^{\gamma(m,n)}\|\phi\|(1+\delta\mu),$$

where

$$\mu = \frac{(2De^{ar})^{q+1}e^{\varepsilon-a}}{1-e^{qa+\varepsilon}}. \tag{15}$$

Hence, in view of (13) we have

$$\|Rx\| \leqslant De^{ar}\|\phi\|(1+\delta\mu) \leqslant 2De^{ar}\|\phi\|,$$

taking δ sufficiently small so that $\delta\mu < 1$. Therefore, $R(\mathcal{C}) \subset \mathcal{C}$. We now show that R is a contraction. By (14) and since $qa+\varepsilon<0$ we have

$$\begin{aligned}
&\left|(Rx)(m+j)-(Ry)(m+j)\right| \\
&\quad \leqslant \sum_{k=n}^{m-1} \left\|T(m+j,k+1)\right\| \cdot \left|f_k(x_k)-f_k(y_k)\right| \\
&\quad \leqslant D\delta \sum_{k=n}^{m-1} e^{a(m+j-k-1)+\varepsilon(k+1)}\|x_k-y_k\|\left(\|x_k\|^q+\|y_k\|^q\right)
\end{aligned}$$

$$\leqslant 2D\big(2De^{ar}\big)^q\delta\|\phi\|^q\|x-y\|\sum_{k=n}^{m-1}e^{(q+1)\gamma(k,n)}e^{a(m+j-k-1)+\varepsilon(k+1)}$$

$$\leqslant 2D\big(2De^{ar}\big)^q\delta e^{a(m+j-n)}e^{\varepsilon-a}\|x-y\|\sum_{k=n}^{m-1}e^{(qa+\varepsilon)(k-n)}$$

$$\leqslant \big(2De^{ar}\big)^{q+1}\delta e^{\gamma(m,n)}\frac{e^{\varepsilon-a}}{1-e^{qa+\varepsilon}}\|x-y\|.$$

We thus obtain

$$\|Rx-Ry\|\leqslant\delta\mu\|x-y\|,$$

with μ as in (15), and R is a contraction in the complete metric space $\mathcal{C}$. Hence, R has a unique fixed point in $\mathcal{C}$, thus satisfying (9). This completes the proof of the theorem. □

3. Existence of stable manifolds

In this section we establish the existence of local stable manifolds for the delay difference equation (7) assuming that the linear equation (3) admits a nonuniform exponential dichotomy. The Lipschitz dependence of the stable manifolds on the perturbation is discussed in Section 5.

3.1. Nonuniform exponential dichotomies

Consider linear operators $L_m: X\to Y$ for $m\in\mathbb{N}$. We say that Eq. (3) admits a *nonuniform exponential dichotomy* if:

1. there exist projections $P_n: X\to X$ for $n\in\mathbb{N}$ such that

$$P_mT(m,n)=T(m,n)P_n,\quad m\geqslant n;$$

2. for each $m\geqslant n$ the operator $U(m,n)=Q_mT(m,n)Q_n$ is invertible from $\operatorname{Im}Q_n$ to $\operatorname{Im}Q_m$, where $Q_m=\mathrm{Id}-P_m$;
3. there exist constants $a<0\leqslant b$, $D\geqslant 1$ and $\varepsilon\geqslant 0$ such that for $m\geqslant n$ we have

$$\big\|T(m,n)P_n\big\|\leqslant De^{a(m-n)+\varepsilon n},\tag{16}$$

$$\big\|U(m,n)^{-1}Q_m\big\|\leqslant De^{-b(m-n)+\varepsilon m}.\tag{17}$$

We then write $E_m=\operatorname{Im}P_m$ and $F_m=\operatorname{Im}Q_m$ for each $m\in\mathbb{N}$.

The condition $a<0$ corresponds to the existence of genuine *stable* behavior. We emphasize that we do not need to assume that $b>0$. Analogously, we could consider the case when $a\leqslant 0<b$, that corresponds to the existence of genuine *unstable* behavior, and establish the existence of local unstable manifolds under perturbations, but in view of the simplicity of the exposition we avoid doing this.

An example of nonuniform exponential dichotomy can be obtained in a similar manner to that in Section 2.2. Namely, we assume that with respect to some invariant decomposition $X = P \times Q$ (with P and Q independent of m), writing $\phi = (\psi, \varphi)$ with values in $P \times Q$ we have

$$L_m\phi = (B_m\phi, C_m\varphi),$$

for some linear operators $B_m : P \to \mathbb{R}^p$ and $C_m : Q \to \mathbb{R}^q$ with $p + q = k$. We set $r = 0$, and let B_m and C_m be respectively

$$B_m\phi = e^{-\omega+\alpha m\cos(\pi m)-\alpha(m-1)\cos(\pi(m-1))+\alpha\sin(\pi m)-\alpha\sin(\pi(m-1))}\phi(0),$$
$$V_m\varphi = e^{\omega-\alpha m\cos(\pi m)+\alpha(m-1)\cos(\pi(m-1))-\alpha\sin(\pi m)+\alpha\sin(\pi(m-1))}\varphi(0).$$

One can then proceed in a similar manner to that in Section 2.2 to show that we obtain a nonuniform exponential dichotomy in which no component is a uniform contraction or a uniform expansion (i.e., one cannot take $\varepsilon = 0$ both in (16) and (17)).

3.2. *Existence of stable manifolds*

Given functions $f_m : X \to Y$ for $m \in \mathbb{N}$, we want to construct stable invariant manifolds for the dynamics

$$v(m+1) = L_m v_m + f_m(v_m). \tag{18}$$

As in Section 2.3, we continue to assume that $f_m(0) = 0$ for each $m \in \mathbb{N}$, and that there exist constants $\delta, q > 0$ (independent of m) such that (8) holds for every $m \in \mathbb{N}$ and $u, v \in X$.

For each $n \in \mathbb{N}$ and $\phi \in X$, we obtain a unique function $v : [n + r, +\infty) \to Y$, denoted by $v(\cdot, n, \phi)$, such that $v_n = \phi$ and (18) holds for all $m \geqslant n$. For each $m \geqslant n$ we define the operator $\mathcal{F}(m, n)$ on X by

$$\mathcal{F}(m, n)\phi = v_m(\cdot, n, \phi), \quad \phi \in X. \tag{19}$$

For each $m \in \mathbb{N}$ and $\beta, \delta > 0$ we set

$$Z_{m,\beta} = Z_{m,\beta}(\delta) = \bigl\{x \in E_m\colon \bigl|x(j)\bigr| \leqslant \delta e^{-\beta m} \text{ for } j \in [r, 0]\bigr\}.$$

Let $\mathcal{X}_\beta$ be the space of sequences $(\phi_m)_{m\in\mathbb{N}}$ of functions $\phi_m : Z_{m,\beta} \to F_m$ such that for each $m \in \mathbb{N}$, $\phi_m(0) = 0$ and

$$\bigl\|\phi_m(x) - \phi_m(y)\bigr\| \leqslant \|x - y\| \quad \text{for every } x, y \in Z_{m,\beta}, \tag{20}$$

using the norm $\|\cdot\|$ in (2). Given $(\phi_m)_{m\in\mathbb{N}} \in \mathcal{X}_\beta$, for each $m \in \mathbb{N}$ we consider the graph

$$\mathcal{V}_m = \bigl\{\bigl(\xi, \phi_m(\xi)\bigr)\colon \xi \in Z_{m,\beta}\bigr\}.$$

The following is our stable manifold theorem. We set $\beta = (1 + 2/q)\varepsilon$.

Theorem 2. *If Eq.* (3) *admits a nonuniform exponential dichotomy, and the conditions*

$$a+\beta \leqslant 0 \quad \textit{and} \quad a+\varepsilon < b \tag{21}$$

are satisfied, then there exist $\delta > 0$ and a unique $\phi \in \mathcal{X}_\beta$ such that

$$\mathcal{F}(m,n)\big(\xi, \phi_n(\xi)\big) \subset \mathcal{V}_m \tag{22}$$

for every $m \geqslant n$ and $\xi \in Z_{n,\beta+\varepsilon}(\delta/(2De^{ar}))$. Furthermore,

$$\big\|\mathcal{F}(m,n)\big(\xi, \phi_n(\xi)\big) - \mathcal{F}(m,n)\big(\bar\xi, \phi_n(\bar\xi)\big)\big\| \leqslant 6De^{a(m-n)+\varepsilon n}\|\xi-\bar\xi\| \tag{23}$$

for every $m \geqslant n$ and $\xi, \bar\xi \in Z_{n,\beta+\varepsilon}$.

4. Proof of Theorem 2

Take $n \in \mathbb{N}$ and write $v_m = (x_m, y_m) \in E_m \times F_m$ for each $m \geqslant n$. The dynamics in (18) satisfies the equations (compare with (10))

$$x_m = T(m,n)x_n + \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}\big(\Gamma f_l(x_l,y_l)\big),$$

$$y_m = T(m,n)y_n + \sum_{l=n}^{m-1} T(m,l+1)Q_{l+1}\big(\Gamma f_l(x_l,y_l)\big).$$

These identities can be obtained in a similar manner to that in Lemma 1. In order that (22) holds we must have

$$x_m = T(m,n)x_n + \sum_{l=n}^{m-1} T(m,l+1)P_{l+1}\big(\Gamma f_l\big(x_l,\phi_l(x_l)\big)\big),$$

$$\phi_m(x_m) = T(m,n)y_n + \sum_{l=n}^{m-1} T(m,l+1)Q_{l+1}\big(\Gamma f_l\big(x_l,\phi_l(x_l)\big)\big). \tag{24}$$

We equip the space $\mathcal{X}_\beta$ with the norm

$$\|\phi\| = \sup\big\{\big\|\phi_m(x)\big\|/\|x\|\colon\ m\in\mathbb{N} \text{ and } x \in Z_{m,\beta}\setminus\{0\}\big\} \tag{25}$$

for each $\phi = (\phi_m)_{m\in\mathbb{N}} \in \mathcal{X}_\beta$. Clearly $\|\phi\| \leqslant 1$, and given $m \in \mathbb{N}$ and $x \neq 0$ we have

$$\big\|\phi_m(x)\big\| \leqslant \delta e^{-\beta m}\big\|\phi_m(x)\big\|/\|x\| \leqslant \delta\|\phi\| \leqslant \delta$$

for every $\phi \in \mathcal{X}_\beta$. This implies that $\mathcal{X}_\beta$ is a complete metric space with the norm in (25). We also set

$$N_\beta = \big\{(m,\xi)\colon\ m \in \mathbb{N} \text{ and } \xi \in Z_{m,\beta}\big\}.$$

In addition, we consider the space $\mathcal{X}_\beta^*$ of sequences $(\phi_m)_{m\in\mathbb{N}}$ of functions $\phi_m : E_m \to F_m$ such that $\phi|N_\beta = (\phi_m|Z_{m,\beta})_{m\in\mathbb{N}} \in \mathcal{X}_\beta$ and for each $m \in \mathbb{N}$,

$$\phi_m(\xi) = \phi_m\big(\delta e^{-\beta m}\xi/\|\xi\|\big) \quad \text{whenever } \xi \notin Z_{m,\beta}.$$

Clearly, $\mathcal{X}_\beta^*$ is also a complete metric space with the norm $\mathcal{X}_\beta^* \ni \phi \mapsto \|\phi|N_\beta\|$.

Lemma 2. *For each $\phi \in \mathcal{X}_\beta^*$ and $m \in \mathbb{N}$ we have*

$$\big\|\phi_m(x) - \phi_m(y)\big\| \leqslant 2\|x-y\| \quad \textit{for every } x, y \in E_m.$$

Proof. In view of (20) we may assume that $x \notin Z_{m,\beta}$. We first consider the case when $y \notin Z_{m,\beta}$. Setting $c = \delta e^{-\beta m}$ we obtain

$$\big\|\phi_m(x) - \phi_m(y)\big\| = \left\|\phi_m\left(c\frac{x}{\|x\|}\right) - \phi_m\left(c\frac{y}{\|y\|}\right)\right\| \leqslant c\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|.$$

Since

$$\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| = \frac{\|(x-y)\|y\| + y(\|y\| - \|x\|)\|}{\|x\|\cdot\|y\|} \leqslant \frac{2\|x-y\|}{\|x\|},$$

we have $\|\phi_m(x) - \phi_m(y)\| \leqslant 2\|x-y\|$. Let now $y \in Z_{m,\beta}$ and take $\kappa \in [0,1)$ such that the vector $z = \kappa x + (1-\kappa)y$ has norm $\|z\| = c$. Then

$$\begin{aligned}\big\|\phi_m(x) - \phi_m(y)\big\| &\leqslant \big\|\phi_m(x) - \phi_m(z)\big\| + \big\|\phi_m(z) - \phi_m(y)\big\| \\ &\leqslant \|x-z\| + 2\|z-y\| \\ &= \|x-y\| + \|z-y\| \leqslant 2\|x-y\|.\end{aligned}$$

This completes the proof of the lemma. $\square$

Lemma 3. *For each $\delta > 0$ sufficiently small and $\phi \in \mathcal{X}_\beta^*$, given $(n,\xi) \in N_\beta$ there is a unique solution of the first identity in* (24) *with $x_n = \xi$. Furthermore,*

$$\|x_m\| \leqslant 2De^{ar}e^{a(m-n)+\varepsilon n}\|\xi\| \quad \textit{for every } m \geqslant n. \tag{26}$$

Proof. Given $\delta > 0$ and $n \in \mathbb{N}$, we consider the space

$$\mathcal{D} = \big\{x : [n+r, +\infty) \to Y\colon \|x\|' \leqslant e^{-\beta n}\big\}$$

with the norm

$$\|x\|' = \big(2De^{ar}\big)^{-1}\sup\big\{\|x_m\|e^{-\gamma(m,n)}\colon m \geqslant n\big\} \tag{27}$$

where $\gamma(m,n)$ was introduced in (13). We note that $\mathcal{D}$ is a complete metric space with the norm in (27).

We now define the operator J on $\mathcal{D}$ by

$$(Jx)_m = \sum_{l=n}^{m-1} T(m, l+1) P_{l+1}\big(\Gamma f_l\big(x_l, \phi_l(x_l)\big)\big),$$

or equivalently by (compare to (14))

$$(Jx)(m+j) = \sum_{l=n}^{m-1} T(m+j, l+1) P_{l+1}\big(\Gamma f_l\big(x_l, \phi_l(x_l)\big)\big)(0),$$

for each $x \in \mathcal{D}$, $m \geqslant n$, and $j \in [r, 0]$. Note that since $\phi \in \mathcal{X}_\beta^*$ we can always compute $\phi_l(x_l)$. Given $x, y \in \mathcal{D}$ and $l \geqslant n$ it follows from Lemma 2 that

$$\big\|\big(x_l, \phi_l(x_l)\big)\big\| = \big\|\big(x_l, \phi_l(x_l) - \phi_l(0)\big)\big\| \leqslant 3\|x_l\| \tag{28}$$

and

$$\big\|\big(x_l, \phi_l(x_l)\big) - \big(y_l, \phi_l(y_l)\big)\big\| \leqslant 3\|x_l - y_l\|.$$

Therefore, by (8) we obtain

$$\begin{aligned} A &:= \big|f_l\big(x_l, \phi_l(x_l)\big) - f_l\big(y_l, \phi_l(y_l)\big)\big| \leqslant 3^{q+1}\delta\|x_l - y_l\|\big(\|x_l\|^q + \|y_l\|^q\big) \\ &\leqslant 2^{q+2}3^{q+1}D^{1+q}e^{ar(q+1)}\delta e^{a(q+1)(l-n)-\varepsilon n}\|x - y\|'. \end{aligned} \tag{29}$$

It follows from (16) that

$$\begin{aligned} &\big|(Jx)(m+j) - (Jy)(m+j)\big| \leqslant \sum_{l=n}^{m-1} \big\|T(m+j, l+1)P_{l+1}\big\| A \\ &\leqslant 2^{q+2}3^{q+1}D^{2+q}e^{ar(q+1)}\delta\|x - y\|' \sum_{l=n}^{m-1} e^{a(m+j-l-1)+\varepsilon(l-1)} e^{a(q+1)(l-n)-\varepsilon n} \\ &\leqslant 2^{q+2}3^{q+1}D^{2+q}e^{ar(q+1)}\delta e^{a(r-1)-\varepsilon}\|x - y\|' e^{a(m-n)} \sum_{l=n}^{m-1} e^{(qa+\varepsilon)(l-n)}. \end{aligned}$$

By (21) we have $qa + \varepsilon < 0$ and thus,

$$\|Jx - Jy\|' \leqslant \theta\|x - y\|', \tag{30}$$

where

$$\theta = 6^{q+1}D^{1+q}e^{arq}\delta e^{a(r-1)-\varepsilon}/\big(1 - e^{qa+\varepsilon}\big).$$

We now choose $\delta > 0$ sufficiently small so that $\theta < 1/2$, and we consider the operator $\bar{J}$ on the space $\mathcal{D}$ defined by

$$(\bar{J}x)(m+j) = T(m+j,n)P_n\xi + (Jx)(m+j), \quad m \geqslant n, \ j \in [r,0].$$

For $y = 0 \in \mathcal{D}$ we obtain $Jy = 0$ (note that $\phi_m(0) = 0$ for every $m \in \mathbb{N}$), and thus, by (30), we have that $\|Jx\|' \leqslant \theta\|x\|'$. On the other hand, by (16) we obtain $\|T(\cdot,n)P_n\xi\|' \leqslant \|\xi\|/2$, and hence

$$\|\bar{J}x\|' \leqslant \|T(\cdot,n)P_n\xi\|' + \|Jx\|' \leqslant \frac{1}{2}\|\xi\| + \theta\|x\|' \leqslant e^{-\beta n}.$$

Therefore, $\bar{J} : \mathcal{D} \to \mathcal{D}$ is well defined. In view of (30) we have

$$\|\bar{J}x - \bar{J}y\|' = \|Jx - Jy\|' \leqslant \theta\|x-y\|',$$

and $\bar{J}$ is a contraction. Thus, there exists a unique $x = x_\phi \in \mathcal{D}$ such that $\bar{J}x = x$. This is equivalent to the first statement in the lemma. To establish the inequality in (26) we note that

$$x_m = \lim_{n\to\infty} (\bar{J}^n 0)_m = \sum_{n=0}^{\infty} (J^n y)_m,$$

where $y = T(\cdot,n)P_n\xi$. Therefore,

$$\|x\|' = \sum_{n=0}^{\infty} \|J^n y\|' \leqslant \sum_{n=0}^{\infty} \frac{1}{2^n}\|y\|' \leqslant \|\xi\|.$$

This completes the proof. $\square$

Given $\phi \in \mathcal{X}_\beta^*$ and $(n,\xi), (n,\bar{\xi}) \in N_\beta$ we denote by x and $\bar{x}$ the unique solutions of the first identity in (24) given by Lemma 3 respectively with $x_n = \xi$ and $\bar{x}_n = \bar{\xi}$.

Lemma 4. *For each $\delta > 0$ sufficiently small, given $\phi \in \mathcal{X}_\beta^*$ and $(n,\xi), (n,\bar{\xi}) \in N_\beta$ we have*

$$\|x_m - \bar{x}_m\| \leqslant 2De^{ar}e^{\gamma(m,n)}\|\xi - \bar{\xi}\| \quad \textit{for every } m \geqslant n. \tag{31}$$

Proof. Take $l \geqslant n$. Proceeding as in (28)–(29) and using (26) in Lemma 3 we obtain

$$\begin{aligned} \left|f_l(x_l, \phi_l(x_l)) - f_l(\bar{x}_l, \phi_l(\bar{x}_l))\right| &\leqslant 3^{q+1}\delta\|x_l - \bar{x}_l\|\left(\|x_l\|^q + \|\bar{x}_l\|^q\right) \\ &\leqslant \eta e^{qa(l-n)-2\varepsilon n}\|x_l - \bar{x}_l\|, \end{aligned} \tag{32}$$

where $\eta = 6^{q+1}D^q e^{arq}\delta^q$. Set

$$\rho_m = \|x_m - \bar{x}_m\| \quad \text{and} \quad T_m = e^{-a(m-n)}\rho_m.$$

Using (16), it follows from (24) that

$$\begin{aligned}\rho_m &\leqslant De^{a(m+r-n)+\varepsilon n}\|\xi-\bar\xi\| + D\eta e^{-a+(1-n)\varepsilon}\sum_{l=n}^{m-1} e^{a(m+r-l)+(qa+\varepsilon)(l-n)}\rho_l \\ &\leqslant De^{ar}e^{a(m-n)}\left(e^{\varepsilon n}\|\xi-\bar\xi\| + \eta e^{-a}\sum_{l=n}^{m-1} e^{-a(l-n)}e^{(qa+\varepsilon)(l-n)}\rho_l\right).\end{aligned}$$

Therefore,

$$T_m \leqslant De^{ar}\left(e^{\varepsilon n}\|\xi-\bar\xi\| + \eta e^{-a}\sum_{l=n}^{m-1} e^{(qa+\varepsilon)(l-n)}T_l\right).$$

Setting $T=\sup_{m\in\mathbb{N}} T_m$, provided that δ is sufficiently small we obtain

$$T \leqslant De^{ar}\left(e^{\varepsilon n}\|\xi-\bar\xi\| + \frac{\eta e^{-a}T}{1-e^{qa+\varepsilon}}\right) \leqslant De^{ar}e^{\varepsilon n}\|\xi-\bar\xi\| + \frac{T}{2}.$$

This establishes the desired statement. $\square$

Given $\phi,\psi\in\mathcal{X}_\beta^*$ and $(n,\xi)\in N_\beta$, we denote by x and y the solutions of the first identity in (24) given by Lemma 3 with $x_n=y_n=\xi$.

Lemma 5. *For each $\delta>0$ sufficiently small, given $\phi,\psi\in\mathcal{X}_\beta^*$ and $(n,\xi)\in N_\beta$ we have*

$$\|x_m-y_m\| \leqslant 2De^{ar}e^{\gamma(m,n)}\|\xi\|\cdot\|\phi-\psi\| \quad \textit{for every } m\geqslant n. \tag{33}$$

Proof. Proceeding as in (29), it follows from Lemmas 2 and 3 that for $l\geqslant n$,

$$\begin{aligned}\bigl|f_l\bigl(x_l,\phi_l(x_l)\bigr)-f_l\bigl(x_l,\psi_l(y_l)\bigr)\bigr| &\leqslant 3^q\delta\bigl\|\bigl(x_l-y_l,\phi_l(x_l)-\psi_l(y_l)\bigr)\bigr\|\bigl(\|x_l\|^q+\|y_l\|^q\bigr) \\ &\leqslant \bar\eta e^{qa(l-n)-2\varepsilon n}\bigl(\|x_l\|\cdot\|\phi-\psi\|+3\|x_l-y_l\|\bigr),\end{aligned} \tag{34}$$

where $\bar\eta = 2\cdot 6^q D^q e^{aqr}\delta^q$. Set

$$\bar\rho_m = \|x_m-y_m\| \quad\text{and}\quad \bar T_m = e^{-a(m-n)}\bar\rho_m.$$

Using (16) and Lemma 3 it follows from (24) that

$$\begin{aligned}\bar T_m &\leqslant D\bar\eta e^{-a(m-n)}\sum_{l=n}^{m-1} e^{a(m+r-l-1)+\varepsilon(l+1)+qa(l-n)-2\varepsilon n}\bigl(\|x_l\|\cdot\|\phi-\psi\|+3\bar\rho_l\bigr) \\ &\leqslant D\bar\eta e^{a(r-1)+(1-n)\varepsilon}2De^{ar}e^{\varepsilon n}\|\xi\|\cdot\|\phi-\psi\|\sum_{l=n}^{m-1} e^{\kappa(l-n)} \\ &\quad + D\bar\eta e^{a(r-1)+(1-n)\varepsilon}3\sum_{l=n}^{m-1} e^{\kappa(l-n)}\bar T_l,\end{aligned}$$

where $\kappa = (q+1)a + \varepsilon < 0$ (see (21)). Setting $\bar{T} = \sup_{m\in\mathbb{N}} \bar{T}_m$, we obtain

$$\bar{T} \leqslant \frac{D\bar{\eta}e^{a(r-1)}}{1-e^{\kappa}}\bigl(2De^{ar}e^{\varepsilon n}\|\xi\| \cdot \|\phi-\psi\| + 3\bar{T}\bigr),$$

and taking δ so small that $D\bar{\eta}e^{a(r-1)}/(1-e^{\kappa}) \leqslant 1/4$ yields

$$\bar{T} \leqslant \frac{1}{2}De^{ar}e^{\varepsilon n}\|\xi\| \cdot \|\phi-\psi\| + \frac{3\bar{T}}{4}.$$

This completes the proof. □

Lemma 6. *Given $\delta > 0$ sufficiently small, there is a unique $\phi \in \mathfrak{X}_\beta^*$ such that for every $(n,\xi) \in N_\beta$ we have*

$$\phi_n(\xi) = -\sum_{l=n}^{\infty} U(l+1,n)^{-1}Q_{l+1}\bigl(\Gamma f_l\bigl(x_l,\phi_l(x_l)\bigr)\bigr). \tag{35}$$

Proof. Set $z_l(\xi) = f_l(x_l,\phi_l(x_l))$, where $(x_l)_{l\geqslant n}$ is the sequence given by Lemma 3 with $x_n = \xi$. We look for a fixed point of the operator Φ defined for each $\phi \in \mathfrak{X}_\beta^*$ by

$$(\Phi\phi)_n(\xi) = -\sum_{l=n}^{\infty} U(l+1,n)^{-1}Q_{l+1}\bigl(\Gamma z_l(\xi)\bigr) \tag{36}$$

whenever $(n,\xi) \in N_\beta$, and otherwise by

$$(\Phi\phi)_n(\xi) = (\Phi\phi)_n\bigl(\delta e^{-\beta n}\xi/\|\xi\|\bigr).$$

We start by showing that the series in (36) converges. By Lemma 3 and (32) we have

$$\bigl|z_l(\xi)\bigr| \leqslant 2\eta De^{ar}\delta e^{(q+1)a(l-n)-2(1+1/q)\varepsilon n}.$$

It follows from (17) that

$$\sum_{l=n}^{\infty}\bigl\|U(l+1,n)^{-1}Q_{l+1}\bigr\| \cdot \bigl|z_l(\xi)\bigr| \leqslant 2\eta De^{ar}\delta e^{-b+\varepsilon(1-n(1+2/q))}\sum_{l=n}^{\infty} e^{(T+qa)(l-n)}$$

where $T = a - b + \varepsilon < 0$ (in view of (21)). Since $a < 0$, we have $T + aq < 0$ and the series converges.

When $\xi = 0$, we have $x_m = 0$ for every $\phi \in \mathfrak{X}_\beta^*$ and $m \geqslant n$. Thus, $(\Phi\phi)_m(0) = 0$ for all $m \geqslant n$. Given $\phi \in \mathfrak{X}_\beta^*$, and (n,ξ), $(n,\bar{\xi}) \in N_\beta$ we now consider the sequences x and $\bar{x}$ given by Lemma 3 with $x_n = \xi$ and $\bar{x}_n = \bar{\xi}$. By (32) and (31) we obtain

$$c_l := \bigl\|z_l(\xi) - z_l(\bar{\xi})\bigr\| \leqslant 2\eta De^{ar}e^{(q+1)a(l-n)-\varepsilon n}\|\xi - \bar{\xi}\|.$$

Using (17), we conclude that

$$\begin{aligned}\big\|(\Phi\phi)_n(\xi)-(\Phi\phi)_n(\bar\xi)\big\| &\leqslant \sum_{l=n}^{\infty}\big\|U(l+1,n)^{-1}Q_{l+1}\big\|c_l\\ &\leqslant 2\eta D^2e^{ar}e^{-b+\varepsilon}\sum_{l=n}^{\infty}e^{(T+qa)(l-n)}\|\xi-\bar\xi\|\\ &=\frac{2\eta D^2e^{ar}e^{-b+\varepsilon}}{1-e^{T+qa}}\|\xi-\bar\xi\|. \end{aligned}\tag{37}$$

Taking $\delta>0$ sufficiently small (so that η is sufficiently small), we obtain

$$\big\|(\Phi\phi)_n(\xi)-(\Phi\phi)_n(\bar\xi)\big\|\leqslant\|\xi-\bar\xi\|$$

for every $\xi,\bar\xi\in Z_{n,\beta}$. This shows that $\Phi(\mathfrak{X}^*_\beta)\subset\mathfrak{X}^*_\beta$.

We now show that the operator $\Phi:\mathfrak{X}^*_\beta\to\mathfrak{X}^*_\beta$ is a contraction. By (34) and (33) we obtain

$$d_l:=\big|f_l\big(x_l,\phi_l(x_l)\big)-f_l\big(y_l,\psi_l(y_l)\big)\big|\leqslant 8\bar\eta D^2e^{ar}e^{(q+1)a(l-n)-\varepsilon n}\|\xi\|\cdot\|\phi-\psi\|.$$

Proceeding as in (37), we conclude that

$$\begin{aligned}\big\|(\Phi\phi)_n(\xi)-(\Phi\psi)_n(\xi)\big\| &\leqslant\sum_{n=0}^{\infty}\big\|U(l+1,n)^{-1}Q_{l+1}\big\|d_l\\ &\leqslant\frac{8\bar\eta D^2e^{ar}e^{-b+\varepsilon}}{1-e^{T+qa}}\|\xi\|\cdot\|\phi-\psi\|.\end{aligned}$$

Therefore, for $\delta>0$ sufficiently small the operator $\Phi:\mathfrak{X}^*_\beta\to\mathfrak{X}^*_\beta$ is a contraction in the complete metric space $\mathfrak{X}^*_\beta$ (see (25)). Hence, there exists a unique $\phi\in\mathfrak{X}^*_\beta$ satisfying $\Phi\phi=\phi$. □

Proof of Theorem 2. Let $\phi\in\mathfrak{X}^*_\beta$ be the unique function given by Lemma 6 such that (35) holds for every $(n,\xi)\in N_\beta$. Using the identity

$$T(m,n)U(l+1,n)^{-1}=T(m,l+1)Q_{l+1},$$

we obtain from (35) that

$$T(m,n)\phi_n(\xi)=-\sum_{l=n}^{\infty}T(m,l+1)Q_{l+1}\big(\Gamma f_l\big(x_l,\phi_l(x_l)\big)\big).$$

Hence, for each $m\geqslant n$,

$$\begin{aligned}&T(m,n)\phi_n(\xi)+\sum_{l=n}^{m-1}T(m,l+1)Q_{l+1}\big(\Gamma f_l\big(x_l,\phi_l(x_l)\big)\big)\\ &\quad=-\sum_{l=m}^{\infty}T(m,l+1)Q_{l+1}\big(\Gamma f_l\big(x_l,\phi_l(x_l)\big)\big).\end{aligned}\tag{38}$$

Moreover, since $a+\beta \leqslant 0$ (see (21)), for $\xi \in Z_{m,\beta+\varepsilon}(\delta/(2De^{ar}))$ we have

$$\|x_m\| \leqslant 2De^{\gamma(m,n)}\|\xi\| \leqslant e^{\gamma(m,n)}\delta e^{-(\beta+\varepsilon)n} \leqslant \delta e^{-\beta m},$$

and $(m, x_m) \in N_\beta$. It follows from (35) that the series in (38) is equal to $\phi_m(x_m)$. To establish the inequality in (23), we note that by Lemma 4,

$$\begin{aligned}\big\|\big(x_m, \phi_m(x_m)\big) - \big(\bar{x}_m, \phi_m(\bar{x}_m)\big)\big\| &\leqslant 3\|x_m - \bar{x}_m\| \\ &\leqslant 6De^{ar}e^{\gamma(m,n)}\|\xi - \bar{\xi}\|\end{aligned}$$

for every $m \geqslant n$. This completes the proof of the theorem. □

5. Behavior under perturbations

We describe in this section how the stable manifolds in Theorem 2 vary with the functions f_m. We consider:

1. linear operators $L_m : X \to Y$ for $m \in \mathbb{N}$;
2. functions $f_m, \bar{f}_m : X \to Y$ for $m \in \mathbb{N}$, with $f_m(0) = \bar{f}_m(0) = 0$, and constants $c, q > 0$ such that (8) holds, also with f_m replaced by $\bar{f}_m$.

Under the hypotheses of Theorem 2, there exist unique sequences $(\phi_m)_{m\in\mathbb{N}}, (\bar{\phi}_m)_{m\in\mathbb{N}} \in \mathfrak{X}^*_\beta$ associated respectively to the maps f_m and $\bar{f}_m$. In addition to defining $\mathcal{F}(m,n)$ by (19), we also consider the dynamics

$$\bar{v}(m+1) = L_m\bar{v}_m + \bar{f}_m(\bar{v}_m),$$

and the operator $\bar{\mathcal{F}}(m,n)$ on X defined by

$$\bar{\mathcal{F}}(m,n)\phi = \bar{v}_m(\cdot, n, \phi), \quad \phi \in X.$$

We will use the notations

$$\|f - \bar{f}\| = \sup\left\{\frac{|f_n(x) - \bar{f}_n(x)|}{\|x\|}e^{2\beta n} \colon n \in \mathbb{N} \text{ and } x \in X \setminus \{0\}\right\}$$

and

$$\|\phi - \bar{\phi}\| = \sup\left\{\frac{\|\phi_m(x_m) - \bar{\phi}_m(x_m)\|}{\|x_m\|}, \frac{\|\phi_m(\bar{x}_m) - \bar{\phi}_m(\bar{x}_m)\|}{\|\bar{x}_m\|}\right\},$$

with the supremum taken over all $(n,\xi) \in N_\beta$, where the sequences $(x_m)_{m\geqslant n}$ and $(\bar{x}_m)_{m\geqslant n}$ are given by Lemma 3 respectively for the sequences $(f_m)_{m\in\mathbb{N}}$ and $(\bar{f}_m)_{m\in\mathbb{N}}$, and the pair (n,ξ).

Theorem 3. *If Eq.* (3) *admits a nonuniform exponential dichotomy, and* (21) *holds, then there exists* $\zeta > 0$ *such that for any sufficiently small* $\delta > 0$ *and* $(n, \xi) \in N_\beta$ *we have*

$$\|\phi - \bar{\phi}\| \leqslant \frac{\zeta\delta}{1 - \zeta\delta^q}\|f - \bar{f}\|.$$

Proof. By Lemma 6, for $(n, \xi) \in N_\beta$ we have

$$\begin{aligned} \phi_n(\xi) &= -\sum_{l=n}^{\infty} U(l+1, n)^{-1} Q_{l+1}\big(\Gamma f_l(z_l)\big), \\ \bar{\phi}_n(\xi) &= -\sum_{l=n}^{\infty} U(l+1, n)^{-1} Q_{l+1}\big(\Gamma \bar{f}_l(\bar{z}_l)\big), \end{aligned} \tag{39}$$

where

$$z_m = \big(x_m, \phi_m(x_m)\big), \qquad \bar{z}_m = \big(\bar{x}_m, \bar{\phi}_m(\bar{x}_m)\big). \tag{40}$$

Using (39) and (8) we obtain

$$\begin{aligned} \big\|\phi_n(\xi) - \bar{\phi}_n(\xi)\big\| &\leqslant \sum_{l=n}^{\infty} \big\|U(l+1, n)^{-1} Q_{l+1}\big\| \cdot \big|f_l(z_l) - \bar{f}_l(z_l)\big| \\ &\quad + \sum_{l=n}^{\infty} \big\|U(l+1, n)^{-1} Q_{l+1}\big\| \cdot \big|\bar{f}_l(z_l) - \bar{f}_l(\bar{z}_l)\big| \\ &\leqslant \|f - \bar{f}\| \sum_{l=n}^{\infty} \big\|U(l+1, n)^{-1} Q_{l+1}\big\| \cdot \|z_l\| e^{-2\beta l} \\ &\quad + \delta \sum_{l=n}^{\infty} \big\|U(l+1, n)^{-1} Q_{l+1}\big\| \cdot \|z_l - \bar{z}_l\| \big(\|z_l\|^q + \|\bar{z}_l\|^q\big). \end{aligned} \tag{41}$$

Notice that by Lemma 3 and (26),

$$\|z_l\| \leqslant 3\|x_l\| \leqslant 6De^{ar} e^{a(l-n)+\varepsilon n}\|\xi\|.$$

Therefore, using (17), for $\xi \in Z_{n,\beta} \setminus \{0\}$ we have

$$\begin{aligned} &\sum_{l=n}^{\infty} \big\|U(l+1, n)^{-1} Q_{l+1}\big\| \cdot \frac{\|z_l\|}{\|\xi\|} e^{-2\beta l} \\ &\quad \leqslant 6D^2 e^{ar} \sum_{l=n}^{\infty} e^{-b(l+1-n)+\varepsilon(l+1)+a(l-n)+\varepsilon n} e^{-2(1+1/q)\varepsilon l} \\ &\quad \leqslant 6D^2 e^{ar} e^{-b+\varepsilon} \sum_{l=n}^{\infty} e^{(a-b)(l-n)} e^{\varepsilon(l+n)-2(1+1/q)\varepsilon l} \end{aligned}$$

$$\leqslant 6D^2 e^{ar} e^{-b+\varepsilon} \sum_{l=n}^{\infty} e^{(a-b)(l-n)} \leqslant \frac{6D^2 e^{ar} e^{\varepsilon}}{1-e^{a-b}}. \tag{42}$$

On the other hand, by (40) and Lemma 2,

$$\begin{aligned}
\|z_l - \bar{z}_l\| &\leqslant \|x_l - \bar{x}_l\| + \big\|\phi_l(x_l) - \bar{\phi}_l(\bar{x}_l)\big\| \\
&\leqslant \|x_l - \bar{x}_l\| + \big\|\phi_l(x_l) - \phi_l(\bar{x}_l)\big\| + \big\|\phi_l(\bar{x}_l) - \bar{\phi}_l(\bar{x}_l)\big\| \\
&\leqslant 3\|x_l - \bar{x}_l\| + \big\|\phi_l(\bar{x}_l) - \bar{\phi}_l(\bar{x}_l)\big\|.
\end{aligned} \tag{43}$$

Lemma 7. *There exists $K > 0$ such that for δ sufficiently small and $l \geqslant n$,*

$$\|x_l - \bar{x}_l\| \leqslant K e^{ar} e^{a(l-n)} \|\xi\| \cdot \|f - \bar{f}\| + \frac{2D}{3} e^{ar} e^{a(l-n)} \|\xi\| \cdot \|\phi - \bar{\phi}\|.$$

Proof of the lemma. Take $l \geqslant n$. By (8) and Lemma 2 we have

$$\big|f_l\big(x_l, \phi_l(x_l)\big) - f_l\big(\bar{x}_l, \bar{\phi}_l(\bar{x}_l)\big)\big| \leqslant 3^q \delta \|z_l - \bar{z}_l\| \big(\|x_l\|^q + \|\bar{x}_l\|^q\big).$$

By (43) and (26), for $\xi \in Z_{n,\beta}$ we obtain

$$\begin{aligned}
&\big|f_l\big(x_l, \phi_l(x_l)\big) - f_l\big(\bar{x}_l, \bar{\phi}_l(\bar{x}_l)\big)\big| \\
&\quad \leqslant 2 \cdot 6^q \delta D^q e^{qar} e^{qa(l-n)+\varepsilon qn} \|\xi\|^q \big(3\|x_l - \bar{x}_l\| + \|\bar{x}_l\| \cdot \|\phi - \bar{\phi}\|\big) \\
&\quad \leqslant \bar{\eta} e^{qa(l-n)-2\varepsilon n} \big(3\|x_l - \bar{x}_l\| + \|\bar{x}_l\| \cdot \|\phi - \bar{\phi}\|\big),
\end{aligned} \tag{44}$$

where $\bar{\eta} = 2 \cdot 6^q D^q e^{(q+1)ar} \delta^q$. We can easily verify that for $m \geqslant n$,

$$\begin{aligned}
x_m &= T(m,n)\xi + \sum_{l=n}^{m-1} T(m, l+1) P_{l+1}\big(\Gamma f_l\big(x_l, \phi_l(x_l)\big)\big), \\
\bar{x}_m &= T(m,n)\xi + \sum_{l=n}^{m-1} T(m, l+1) P_{l+1}\big(\Gamma \bar{f}_l\big(\bar{x}_l, \bar{\phi}_l(\bar{x}_l)\big)\big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|x_m - \bar{x}_m\| \leqslant &\sum_{l=n}^{m-1} \big\|T(m, l+1) P_{l+1}\big\| \cdot \big|f_l\big(x_l, \phi_l(x_l)\big) - f_l\big(\bar{x}_l, \bar{\phi}_l(\bar{x}_l)\big)\big| \\
&+ \sum_{l=n}^{m-1} \big\|T(m, l+1) P_{l+1}\big\| \cdot \big|f_l\big(\bar{x}_l, \bar{\phi}_l(x_l)\big) - \bar{f}_l\big(\bar{x}_l, \bar{\phi}_l(\bar{x}_l)\big)\big|.
\end{aligned}$$

Using (16), Lemma 2, and (26) we obtain

$$\begin{aligned}
&6\delta D^2 e^{ar}\|\xi\|\cdot\|f-\bar f\|\sum_{l=n}^{m-1} e^{a(m-l-1)+\varepsilon(l+1)}e^{a(l-n)+\varepsilon n}e^{-2(1+2/q)\varepsilon l}\\
&\quad\leqslant 6\delta D^2 e^{ar}e^{a(m-n-1)+\varepsilon(n+1)}\|\xi\|\cdot\|f-\bar f\|\sum_{l=n}^{m-1} e^{-(1+4/q)\varepsilon l}\\
&\quad\leqslant \frac{6\delta D^2 e^{ar}e^{a(m-n-1)+\varepsilon(1-4n/q)}}{1-e^{-(1+4/q)\varepsilon}}\|\xi\|\cdot\|f-\bar f\|\\
&\quad\leqslant K_0 e^{ar}e^{a(m-n)}\|\xi\|\cdot\|f-\bar f\|,
\end{aligned}$$

where

$$K_0=\frac{6\delta D^2 e^{-a+\varepsilon}}{1-e^{-(1+4/q)\varepsilon}}.$$

Set now

$$\bar\rho_m=\|x_m-\bar x_m\| \quad\text{and}\quad \bar T_m=e^{-a(m-n)}\bar\rho_m.$$

Using (16) and (44), we obtain

$$\begin{aligned}
\bar T_m &\leqslant \bar\eta e^{-a(m-n)}\sum_{l=n}^{m-1} e^{a(m-l-1)+\varepsilon(l+1)+qa(l-n)-2\varepsilon n}\big(3\bar\rho_l+\|\bar x_l\|\cdot\|\phi-\bar\phi\|\big)\\
&\quad+K_0 e^{ar}\|\xi\|\cdot\|f-\bar f\|.
\end{aligned}$$

Furthermore, using (26),

$$\begin{aligned}
\bar T_m &\leqslant \bar\eta e^{-a+\varepsilon}\left(3e^{-\varepsilon n}\sum_{l=n}^{m-1} e^{\kappa(l-n)}\bar T_l+2De^{ar}\|\xi\|\cdot\|\phi-\bar\phi\|\sum_{l=n}^{m-1}e^{\kappa(l-n)}\right)\\
&\quad+K_0 e^{ar}\|\xi\|\cdot\|f-\bar f\|,
\end{aligned}$$

where $\kappa=qa+\varepsilon<0$, in view of (21). Setting $\bar T=\sup_{m\in\mathbb{N}} T_m$ we obtain

$$\bar T\leqslant \frac{\bar\eta e^{-a+\varepsilon}}{1-e^{\kappa}}\big(3\bar T+2De^{ar}\|\xi\|\cdot\|\phi-\bar\phi\|\big)+K_0e^{ar}\|\xi\|\cdot\|f-\bar f\|.$$

Taking δ sufficiently small so that $\bar\eta e^{-a+\varepsilon}/(1-e^{\kappa})\leqslant 1/6$ yields

$$\bar T\leqslant \frac{\bar T}{2}+\frac{D}{3}e^{ar}\|\xi\|\cdot\|\phi-\bar\phi\|+K_0e^{ar}\|\xi\|\cdot\|f-\bar f\|.$$

This completes the proof of the lemma. □

Using now (43), (25), and Lemma 7, we obtain

$$\begin{aligned}\|z_l-\bar z_l\| &\leqslant \bigl(2De^{ar}e^{a(l-n)}\|\xi\|+\|\bar x_l\|\bigr)\|\phi-\bar\phi\| \\ &\quad + 3Ke^{ar}e^{a(l-n)}\|\xi\|\cdot\|f-\bar f\| \\ &\leqslant 4De^{ar}e^{a(l-n)+\varepsilon n}\|\xi\|\cdot\|\phi-\bar\phi\| \\ &\quad + 3Ke^{ar}e^{a(l-n)}\|\xi\|\cdot\|f-\bar f\|. \end{aligned} \tag{45}$$

Finally, by (17), (45), Lemma 2, and (26), for $\xi\in Z_{n,\beta}\setminus\{0\}$ we have

$$\begin{aligned}&\frac{1}{6^qD^{q+1}e^{(q+1)ar}}\sum_{l=n}^{\infty}\bigl\|U(l+1,n)^{-1}Q_{l+1}\bigr\|\cdot\frac{\|z_l-\bar z_l\|}{\|\xi\|}\bigl(\|z_l\|^q+\|\bar z_l\|^q\bigr) \\ &\leqslant 8D\sum_{l=n}^{\infty}e^{-b(l+1-n)+\varepsilon(l+1)}e^{(q+1)a(l-n)+(q+1)\varepsilon n}\|\xi\|^q\|\phi-\bar\phi\| \\ &\quad + 6K\sum_{l=n}^{\infty}e^{-b(l+1-n)+\varepsilon(l+1)}e^{(q+1)a(l-n)+q\varepsilon n}\|\xi\|^q\|f-\bar f\| \\ &\leqslant 8De^{-b+\varepsilon}\delta^q\sum_{l=n}^{\infty}e^{((q+1)a-b+\varepsilon)(l-n)}e^{(q+2)\varepsilon n-\beta qn}\|\phi-\bar\phi\| \\ &\quad + 6Ke^{-b+\varepsilon}\delta^q\sum_{l=n}^{\infty}e^{((q+1)a-b+\varepsilon)(l-n)}e^{(q+1)\varepsilon n-\beta qn}\|f-\bar f\| \\ &\leqslant 8De^{-b+\varepsilon}\delta^q\sum_{l=n}^{\infty}e^{((q+1)a-b+\varepsilon)(l-n)}\|\phi-\bar\phi\| \\ &\quad + 6Ke^{-b+\varepsilon}\delta^q\sum_{l=n}^{\infty}e^{((q+1)a-b+\varepsilon)(l-n)}\|f-\bar f\| \\ &\leqslant \frac{8De^{\varepsilon}\delta^q}{1-e^{(q+1)a-b+\varepsilon}}\|\phi-\bar\phi\|+\frac{6Ke^{\varepsilon}\delta^q}{1-e^{(q+1)a-b+\varepsilon}}\|f-\bar f\|, \end{aligned} \tag{46}$$

since by (21) we have $(q+1)a-b+\varepsilon<0$. By (42) and (46), provided that δ is sufficiently small it follows from (41) that

$$\begin{aligned}\|\phi-\bar\phi\| &\leqslant \delta\left(\frac{6D^2e^{ar}e^{\varepsilon}}{1-e^{a-b}}+\frac{6^{q+1}D^{q+1}e^{(q+1)ar}Ke^{\varepsilon}\delta^q}{1-e^{(q+1)a-b+\varepsilon}}\right)\|f-\bar f\| \\ &\quad + \frac{8\cdot 6^q\delta D^{q+2}e^{(q+1)ar}e^{\varepsilon}\delta^q}{1-e^{(q+1)a-b+\varepsilon}}\|\phi-\bar\phi\|. \end{aligned}$$

This completes the proof of the theorem. □

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